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Rigid Motions Relative to an Observer: *L*-Rigidity

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A new definition of rigidity, L-rigidity, in general relativity is proposed. This concept is a special class of pseudorigid motions and therefore it depends on the chosen curve L. It is shown that, for slow-rotation steady motions in Minkowski space, weak rigidity and L-rigidity are equivalent. The methods of the PPN approximation are considered. In this formalism, the equations that characterize L-rigidity are expressed. As a consequence, the baryon mass density is constant in first order, the stress tensor is constant in the comoving system, the Newtonian potential is constant along the line L, and the gravitational field is constant along the line L in the comoving system.

1. INTRODUCTION

Relativistic rigidity has been the subject of much work practically since the beginning of relativity, starting with Born (1909), Herglotz (1910), and Noether (1910). We will concentrate here on work appearing in the second half of this century, among others, Ehlers and Kundt (1962), Trautmann (1965), Synge (1966, 1972), Dixon (1970, 1979), Ehlers and Rudolph (1977), Köhler and Schattner (1979), Martínez Salas and Gambi (1981), and Del Olmo and Olivert (1983, 1985, 1986, 1987).

Dixon proposed a dynamical criterion for rigidity, in contrast to the Born condition: A body (or the motion of a body) is dynamically rigid if the reduced multipole moments of the momentum-energy tensor have constant components with respect to a comoving orthonormal frame. Dynamical rigidity ensures that the total internal energy is a constant of the motion. This concept is identical to the quasirigidity of Ehlers and Rudolph. In order to point out sufficient conditions under which quasirigidity may be used as an

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approximation, Ehlers and Rudolph defined kinematically a class of pseudorigid motions and compared them with Born rigid motions. Köhler and Schattner analyzed the concept of pseudorigidity to determine whether a given motion is pseudorigid. They showed that under a certain assumption pseudorigidity implies Born rigidity at the center of the motion.

Starting with a simple characterization of pairs of rigidly joined world lines, Martínez Salas and Gambi made successive generalizations in definitions. They connected with the work of Synge, and Ehlers and Kundt, but they did not indicate the connection with the conditions of dynamical rigidity.

Del Olmo and Olivert proposed both dynamical and kinematical criteria for rigidity: An almost-thermodynamic material scheme is weakly rigid if the expansion velocity scalar vanishes and the energy-momentum tensor has constant components in a comoving nonrotating tetrad. Weak rigidity leads to the incompressibility condition given by Ferrando and Olivert (1981). Using the methods of the PNN formalism, they obtained under weak rigidity conditions, an increase of two orders of magnitude in the strain rate tensor.

The initial purpose of this work was to relate weak rigidity with quasirigidity. The attempt failed because it did not yield a direct relation between weak rigidity and quasirigidity. So we introduce a new concept of rigidity, which we call L-rigidity, that in particular cases leads to weak rigidity and quasirigidity. L-Rigidity is a special class of pseudorigid motions and therefore it depends on the chosen curve L. In L-rigid motion, the expansion of the vector field that describes the pseudorigid motion vanishes. Another kinematical condition requires that the Lie derivative of the normal 1-form to the pseudorigid "body" world-tube, with respect to the vector field that describes the pseudorigid motion, vanishes. The third condition is a dynamical one: The family of tensor fields along L obtained by parallely transporting the energy-momentum tensor from the world lines that constitute the pseudorigid motion to the "center of motion" L has constant components with respect to a comoving orthonormal frame.

The study of L-rigidity must begin by showing whether the proposed idealization is approximately satisfied by a wide range of real physical systems. Thus, we study the L-rigidity conditions by applying some techniques of the PPN formalism in general relativity.

In Section 3 we propose (Definition 2) the definition of L-rigidity. We consider a type of pseudorigid motion given by Proposition 1; moreover, this proposition delimits the size of the pseudorigid body locally. Later we express the L-rigidity conditions in a bitensorial form (Synge, 1966; DeWitt and Brehme, 1960). In the same section we use equation (8) together with the definition of the center-of-mass line defined by Ehlers and Rudolph (1977) to obtain, in the Minkowski space-time, that L-rigidity and weak rigidity are equivalent when the angular velocity is both small and constant.

In Section 4 we give the PPN expressions for the L-rigidity equations by using the covariant expansion of bitensors developed by DeWitt and Brehme (1960). The software package Mathematica (Wolfram, 1990) is used to facilitate computation of the PPN L-rigidity equations. The obtained results agree with classical rigidity, in the sense that the baryon mass density is constant in first order [equation (24)] and the stress tensor is constant in the comoving system [equation (25)]. Equations (28) and (29) are also in agreement with classical rigidity because the Newtonian potential is constant along the line L and the gravitational field is constant along the line L in the comoving system.

2. NOTATION

We will consider space-time as a set of three components $(\mathcal{M}, g, \nabla)$, where $\mathcal{M}$ stands for a connected four-dimensional pseudo-Riemannian manifold of Hausdorff type (Sachs and Wu, 1977), g will be a hyperbolic metric tensor field [of signature (3, 1)], and ∇ is the unique linear connection that $\mathcal{M}$ possesses, compatible with g and without torsion.

Absolute differentiation along a curve L, parametrized as $z(s)$, of a vector field X along L is denoted by

$$\dot{X}(s) = \nabla_{\dot{z}(s)}X$$

where $\dot{z}(s)$ is the tangent vector to L at the point $z(s)$.

Symmetrization of indices is denoted by $(\cdot)$.

In each case, Greek labels indicate values of indices from 1 to 4, while Latin labels are used for indices from 1 to 3.

We will use the theory of bitensors developed by Synge (1966) and by DeWitt and Brehme (1960). So, given a bitensor $I^{\lambda\alpha}(z, m)$, we consider that indices represented by Greek letters κ, λ, ... (Latin letters i, j, ...) are associated to the point z, while α, β, ... (a, b, ...) are associated to the point m. We denote by $\sigma(z, m)$ the world function and by

$$H^{\alpha}{}_{\lambda} = (-\sigma_{.}{}^{\lambda}{}_{\alpha})^{-1}, \qquad K^{\alpha}{}_{\lambda} = H^{\alpha}{}_{\mu}\sigma_{.}{}^{\mu}{}_{\lambda}$$

the bitensors introduced by Dixon (1970). In this notation, a dot followed by an index denotes covariant differentiation related to one or another variable.

Another important bitensor that we will use is the parallel propagator, denoted by $\bar{g}_{\lambda\alpha}(z, m)$.

The position vector of m relative to z is denoted by

$$r^{\lambda}(z, m) = \delta^{\lambda}_{\alpha}x^{\alpha}(m) - x^{\lambda}(z)$$

On the other hand, from a set of three components (L, n, Ω), where L is a timelike curve, n is a timelike unit vector field along L, and Ω is a skew tensor field along L orthogonal to n, we can define a transport, rotating M-transport, which may be used to define a derivative operator along L,

$$\overset{\mathrm{RM}}{D}_n X^\lambda = \dot{X}^\lambda + M^\lambda{}_\mu X^\mu$$

where

$$M^\lambda{}_\mu = \dot{n}^\lambda n_\mu - n^\lambda \dot{n}_\mu + \Omega^\lambda{}_\mu$$

This derivative operator is obtained by mixing a transport of Jaumann type (Jaumann, 1911), if $\Omega_{\lambda\mu}$ was the rotation tensor of a timelike congruence, and Dixon's M-transport (Dixon, 1970).

The comoving orthonormal frame used to define the quasirigidity condition satisfies the former transport law (rotating M-transport).

We denote by τ_s: $T_{z(0)}\mathcal{M} \to T_{z(s)}\mathcal{M}$ the induced isomorphism by the rotating M-transport.

3. *L*-RIGIDITY

In the definition of pseudorigid motion, the only limitation that exists concerns the size of the body.

Our interest, first, is centered on delimiting the size of the pseudorigid body explicitly and locally, so that its points are simultaneous in Landau's sense (Olivert, 1980).

Second, we propose a special class of pseudorigid motions, L-rigid motions, and analyze its immediate consequences.

The following result will be used throughout this paper.

Proposition 1. There exists an open interval I of $0 \in \mathrm{R}$ such that for all $s \in I$ there exist convex open sets $N_{z(s)}$ of $0_{z(s)} \in T_{z(s)}\mathcal{M}$ and connected open sets $U_{z(s)}$ of $z(s)$ in $\mathcal{M}$ so that

$$\tau_s(N_{z(0)}) = N_{z(s)}$$

and so that $\exp_{z(s)}$: $N_{z(s)} \to U_{z(s)}$ is a diffeomorphism.

Proof. Given the point $z(0)$, there exists (Kobayashi and Nomizu, 1963) an open neighborhood V of $0_{z(0)} \in T\mathcal{M}$ and an open neighborhood W of $z(0)$ $\in \mathcal{M}$ such that

$$\begin{aligned} \delta\colon\quad & V \to W \times W \\ & X \to (\pi(X), \exp_{\pi(X)} X) \end{aligned}$$

is a diffeomorphism, where π: $T\mathcal{M} \to \mathcal{M}$ is the projection of the tangent fiber bundle. So, for $m \in W$ we have that $\exp_m$: $V_m \to W$, where $V_m = V \cap T_m\mathcal{M}$, is a diffeomorphism. Let $J = L^{-1}(W)$; we consider the differentiable map

$$\tau: \quad J \times T_{z(0)}\mathcal{M} \to T\mathcal{M}$$

$$(s, X) \to \tau_s X$$

Therefore, there exists an open neighborhood I of $0 \in$ R and a convex open neighborhood $N_{z(0)} \subseteq V_{z(0)}$ so that

$$0_{z(s)} \in \tau_s(N_{z(0)}) \subseteq V_{z(s)}, \qquad \forall s \in I$$

Let $N_{z(s)} = \tau_s(N_{z(0)})$ and let $U_{z(s)} = \exp_{z(s)}(N_{z(s)})$. ∎

Because of this proposition, and from the submersion

$$\phi_s: \quad U_{z(s)} \to \mathrm{R}$$

$$m \to \phi_s(m) = (\exp_{z(s)}^{-1}m)^\kappa n_\kappa(s)$$

we trivially obtain that $\Sigma(s) = \phi_s^{-1}(0)$ is a spacelike regular submanifold of dimension 3, called a Landau manifold by Olivert (1980), the points of which are simultaneous in Landau's sense (simultaneity hypersurface). Moreover, as

$$\Sigma(s) = \exp_{z(s)}(N_{z(s)} \cap \tilde{\Sigma}(s))$$

where $\tilde{\Sigma}(s)$ is the orthogonal hyperplane to the vector $n(s)$ at the point $z(s)$, we have that $\Sigma(s)$ is a connected hypersurface.

We denote by $\Sigma = \cup_s\Sigma(s)$ the world tube containing L and by χ the differentiable function on Σ given by (Dixon, 1974)

$$\chi(m) = s \qquad \text{if} \quad m \in \Sigma(s)$$

In order to simplify the notation, we consider

$$s_m = \chi(m), \qquad z_m = z(\chi(m))$$

Likewise, we denote by N^α the unit normal vector field to Σ.

We consider the following congruence of timelike curves:

$$\gamma_m(s) = \exp_{z(s_m+s)}\tau_{s_m+s}\tau_{s_m}^{-1} \exp_{z_m}^{-1}(m) \tag{1}$$

where $m \in \Sigma$, and s belongs to an open neighborhood of the point $-s_m$ given by Proposition 1. The vector field whose integral curves are given by equation (1) has the following components:

$$\omega^\alpha(m) = K^\alpha{}_\kappa(z_m, m)\dot{z}^\kappa(s_m) + H^\alpha{}_\kappa(z_m, m)M^\kappa{}_\lambda(s_m)\sigma_{.}{}^\lambda(z_m, m) \tag{2}$$

whose expression formally coincides with that given by Köhler and Schattner to describe pseudorigid motion. This vector field satisfies

$$\omega^\alpha\chi_{,\alpha} = 1 \tag{3}$$

Obviously Proposition 1 limits the size of the pseudorigid body to the world tube Σ. The same proposition allows us to affirm that the vector field flow that describes pseudorigid motion is a diffeomorphism that maps simultaneity hypersurfaces into simultaneity hypersurfaces, if we consider them sufficiently close, and it preserves the orientation.

From this class of pseudorigid motions, we are going to propose a concept of rigidity that includes a slight modification of the weak rigidity conditions, so that in some particular cases it coincides with the one introduced by Del Olmo and Olivert (1985).

Definition 2. A pseudorigid motion is L-rigid if the following conditions are satisfied:

(i) $\nabla \cdot \omega = 0$

(ii) $(\overset{\mathrm{RM}}{D_n}\overline{T})(s_m, m) = 0, \; \forall m \in \Sigma$

(iii) $\iota_N \iota_N \mathfrak{L}_\omega g = 0$

where

$$\overline{T}^{\lambda\mu}(s, m) = \overline{g}^{\lambda}{}_{\alpha}(z(s), \gamma_m(s - s_m))\overline{g}^{\mu}{}_{\beta}(z(s), \gamma_m(s - s_m)) \times T^{\alpha\beta}(\gamma_m(s - s_m))$$

The last expression indicates that $\overline{T}$ is a family of tensor fields along L obtained by parallely transporting the energy-momentum tensor from the trajectories γ_m ($m \in \Sigma$) to the "center of motion" L.

The first condition of L-rigidity represents the free expansion that every rigid motion must satisfy (as is the case of weakly rigid motion), in our case L-rigid motion. The second one assumes that the family of tensor fields along L, $\overline{T}$, has constant components with respect to a comoving orthonormal frame. This supposes a modification of weak rigidity in the sense that here rotating M-transport is used instead of Fermi–Walker transport.

As for the third condition of L-rigidity, we immediately have that, because N is a unit vector field, it is equivalent to the expression

$$g(\mathfrak{L}_\omega N, N) = 0 \tag{4}$$

Moreover, we get the following result.

Proposition 3. A necessary and sufficient condition for the third condition of L-rigidity is

$$\mathfrak{L}_\omega \iota_N g = 0$$

Proof. First, as N is a unit vector field, we have

$$(\mathfrak{L}_\omega \iota_N g)(N) = -g(N, \mathfrak{L}_\omega N) \tag{5}$$

On the other hand, given $m \in \Sigma$, we denote by $i_{s_m}: \Sigma(s_m) \to \mathcal{M}$ the canonical immersion. As $\Sigma(s_m)$ is a fiber of ϕ_{s_m}, we immediately have

$$i_{s_{m*_m}}(T_m\Sigma(s_m)) = \ker \phi_{s_{m*_m}}$$

and so

$$(\mathfrak{L}_\omega \iota_N g)_m(X) = 0, \qquad \forall X \in \iota_{s_{m*_m}} T_m(\Sigma(s_m))$$

From here and equation (5) we obtain the result. ∎

The latter proposition reaffirms that the vector field ω^α transforms $\Sigma(s)$ into $\Sigma(s + ds)$. Moreover, if ω^α is a Killing vector field, then the third condition of L-rigidity is an identity.

We need to express the three conditions that define L-rigidity in an equivalent way such that the methods of the PPN approximation can be applied.

We begin with the third condition of L-rigidity. On expressing equation (4) in a bitensorial form, we get

$$\begin{aligned} &n_\nu \sigma_{.}{}^{\nu\beta}\{n^\rho \sigma_{.\rho\alpha}[K^\alpha{}_{\kappa.\beta}\dot{z}^\kappa + H^\alpha{}_{\kappa.\beta}M^\kappa{}_\lambda \sigma_{.}{}^\lambda - \dot{n}_\lambda \sigma_{.}{}^\lambda{}_\beta \\ &\quad + (K^\alpha{}_{\kappa.\lambda}\dot{z}^\lambda \dot{z}^\kappa + H^\alpha{}_{\kappa.\lambda}\dot{z}^\lambda M^\kappa{}_\mu \sigma_{.}{}^\mu)\chi_{,\beta}] \\ &\quad - (n_\kappa \dot{M}^\kappa{}_\lambda \sigma_{.}{}^\lambda + n_\lambda \sigma_{.}{}^\lambda{}_\mu \dot{z}^\mu + \dot{n}_\lambda \sigma_{.}{}^\lambda{}_\mu \dot{z}^\mu)\chi_{,\beta}\} = 0 \end{aligned} \tag{6}$$

provided we take into account

$$\begin{aligned} \nabla_\beta \omega^\alpha = {} & K^\alpha{}_{\kappa.\beta}\dot{z}^\kappa + H^\alpha{}_{\kappa.\beta}M^\kappa{}_\lambda \sigma_{.}{}^\lambda + H^\alpha{}_\kappa M^\kappa{}_\lambda \sigma_{.}{}^\lambda{}_\beta + (K^\alpha{}_{\kappa.\lambda}\dot{z}^\lambda \dot{z}^\kappa \\ & + K^\alpha{}_\kappa \ddot{z}^\kappa + H^\alpha{}_{\kappa.\lambda}\dot{z}^\lambda M^\kappa{}_\mu \sigma_{.}{}^\mu + H^\alpha{}_\kappa \dot{M}^\kappa{}_\lambda \sigma_{.}{}^\lambda + H^\alpha{}_\kappa M^\kappa{}_\lambda \sigma_{.}{}^\lambda{}_\mu \dot{z}^\mu)\chi_{,\beta} \end{aligned}$$

Moreover, from this last expression we deduce that the first condition of L-rigidity is equivalent to

$$\begin{aligned} &K^\alpha{}_{\kappa.\alpha}\dot{z}^\kappa + H^\alpha{}_{\kappa.\alpha}M^\kappa{}_\lambda \sigma_{.}{}^\lambda + (K^\alpha{}_{\kappa.\lambda}\dot{z}^\lambda \dot{z}^\kappa + K^\alpha{}_\kappa \ddot{z}^\kappa \\ &\quad + H^\alpha{}_{\kappa.\lambda}\dot{z}^\lambda M^\kappa{}_\mu \sigma_{.}{}^\mu + H^\alpha{}_\kappa \dot{M}^\kappa{}_\lambda \sigma_{.}{}^\lambda + H^\alpha{}_\kappa M^\kappa{}_\lambda \sigma_{.}{}^\lambda{}_\mu \dot{z}^\mu)\chi_{,\alpha} = 0 \end{aligned} \tag{7}$$

Finally, from a direct calculation, the second condition of the L-rigidity leads to

$$\begin{aligned} &2\dot{z}^\nu \bar{g}^{(\lambda}{}_{\alpha.\nu}\bar{g}^{\mu)}{}_\beta T^{\alpha\beta} + 2\omega^\gamma \bar{g}^{(\lambda}{}_{\alpha.\gamma}\bar{g}^{\mu)}{}_\beta T^{\alpha\beta} \\ &\quad + \bar{g}^\lambda{}_\alpha \bar{g}^\mu{}_\beta \nabla_\omega T^{\alpha\beta} + 2\bar{g}^{(\lambda}{}_\alpha M^{\mu)}{}_\rho \bar{g}^\rho{}_\beta T^{\alpha\beta} = 0 \end{aligned} \tag{8}$$

To conclude this section and as an immediate application of this last expression, we study the relation between L-rigidity and weak rigidity. We restrict consideration to Minkowski space. We consider the center-of-mass line L_o defined by Ehlers and Rudolph (1977),

$$S^{\lambda\mu}p_\mu = 0$$

and we take n^λ according to the relation

$$p^\lambda = Mn^\lambda$$

where p^λ and $S^{\lambda\mu}$ are the linear and the angular moments given by Dixon (1974).

With this choice we can show that

$$n^\lambda = \dot{z}^\lambda$$

$$\dot{p}^\lambda = 0 = \dot{S}^{\lambda\mu}$$

which yields that, if $\Omega_{\lambda\mu}$ is the angular velocity of the body, the vector field that determines the pseudorigid motion has the following components:

$$\omega^\alpha = \delta^\alpha_\kappa(\dot{z}^\kappa - \Omega^\kappa{}_\lambda r^\lambda) \tag{9}$$

We consider L-rigid motion with constant angular velocity. The unique nontrivial condition of L-rigidity is, corresponding to (8),

$$\delta^\lambda_\alpha \delta^\mu_\beta \omega^\gamma T^{\alpha\beta}{}_{,\gamma} + 2\delta^{(\lambda}_\alpha \Omega^{\mu)}{}_\nu \delta^\nu_\beta T^{\alpha\beta} = 0 \tag{10}$$

Now, by identifying the kinematical velocity ω^α with the dynamical velocity u^α, we obtain from equation (9) that

$$u^\alpha = \delta^\alpha_\kappa(\dot{z}^\lambda - \Omega^\kappa{}_\lambda r^\lambda) + O(\Omega^2) \tag{11}$$

and therefore equation (10) becomes

$$u^\gamma T^{\alpha\beta}{}_{,\gamma} \sim O(\Omega) \tag{12}$$

On the other hand, the weak rigidity conditions can be written as

$$u^\gamma T^{\alpha\beta}{}_{,\gamma} \sim O(\Omega^2) \tag{13}$$

which indicates that, if the angular velocity is constant and small, weak rigidity and L-rigidity are equivalent.

4. PPN EQUATIONS OF *L*-RIGIDITY. CONSEQUENCES

In this section, we develop the post-Newtonian approximation of (6)–(8) with the idea of obtaining new results that correspond to classical rigidity properties. We will use the notation and results presented by Misner *et al.* (1973).

In this approximation we choose L-rigid motion so that

$$n^i = 0 \tag{14}$$

in the PPN coordinate system. Moreover, we parametrize the curve L with the coordinate time, i.e.,

$$\dot{z}^4 = 1 \tag{15}$$

Equation (14) shows that, as will be shown further on, the simultaneity hypersurfaces coincide, in first order, with the hypersurfaces t = const.

In this approximation, the assumption of small velocities leads to

$$\begin{aligned} \dot{z}^i &\sim O(\epsilon) \\ \Omega_{ij} &\sim O(\epsilon/R) \end{aligned} \tag{16}$$

where ϵ is a small parameter, $\epsilon^2 \sim M/R$, where M and R are typical values of masses and separations of the bodies involved.

On the other hand, as Dixon (1970) indicates, $-\sigma_{.}{}^{\lambda}(z, m)$ is a natural generalization of the position vector of m relative to z, and it is reduced to this position vector in flat space-time. So, in this approximation, we consider that

$$\begin{aligned} \sigma_{.}{}^{\lambda}(z, m) &= -r^{\lambda}(z, m) + A^{\lambda}(z, m) \\ A^{\lambda}(z, m) &\sim O(r^{\lambda}(z, m)\epsilon^2) \end{aligned} \tag{17}$$

We also suppose the following expansion, with respect to the parameter ϵ, of the parallel propagator:

$$\begin{aligned} \bar{g}^i{}_a &= \delta_{ia} + O(\epsilon^2) \\ \bar{g}^i{}_4, \bar{g}^4{}_a &\sim O(\epsilon^3) \\ \bar{g}^4{}_4 &= 1 + O(\epsilon^2) \end{aligned} \tag{18}$$

Obviously, for all $m \in \Sigma$,

$$r^i(z_m, m) \sim O(R) \tag{19}$$

and therefore

$$r^4(z_m, m) \sim O(R\epsilon^3) \tag{20}$$

which indicates that the hypersurface $t = t_m$ coincides, in first order, with the simultaneity hypersurface $\Sigma(t_m)$.

The above result shows that the vector field ω^α that determines the pseudorigid motion has the following components:

$$\begin{aligned} \omega^a(m) &= \delta_i^a \dot{z}^i(t_m) - \delta_i^a \Omega_{ij}(t_m) r^j(z_m, m) + O(\epsilon^3) \\ \omega^4(m) &= 1 + O(\epsilon^2) \end{aligned} \tag{21}$$

Moreover, from (20) it is easy to prove that

$$\chi_{,a} \sim O(\epsilon^3) \tag{22}$$

and from (3)

$$\chi_{,4} = 1 + O(\epsilon^2) \tag{23}$$

From this we get the post-Newtonian expressions of the bitensors that constitute the L-rigidity equations, and using these expressions, we deduce that (8) is equivalent to the following:

$$\frac{d\rho_o}{dt} \sim O\left(\frac{\rho_o \epsilon^3}{R}\right) \tag{24}$$

$$\frac{dt_{\bar{a}\bar{b}}}{dt} + \delta_{ia}\delta_{jc}\Omega_{ij}t_{\bar{c}\bar{b}} + \delta_{jb}\delta_{ic}\Omega_{ji}t_{\bar{a}\bar{c}} \sim O\left(\frac{\rho_o \epsilon^5}{R}\right) \tag{25}$$

Obviously, (24) is in agreement with the classical rigidity, because it supposes that the baryon mass density is constant in first order. Condition (25) indicates that the stress tensor has constant components in the comoving system.

We obtain the post-Newtonian approximation of equation (7):

$$\begin{aligned} U_{,i}\dot{z}^i + U_{,4} = &-(U_{,j4} + \Omega_{jk}U_{,k} + U_{,jk}\dot{z}^k)r^j \\ &+ \tfrac{2}{3}\delta_{ib}\delta_{jd}U_{,bd}\Omega_{ik}r^j r^k + O(\epsilon^5/R) \end{aligned} \tag{26}$$

whereas equation (6) is equivalent to

$$\begin{aligned} U_{,i}\dot{z}^i + U_{,4} = &-(U_{,j4} + \Omega_{jk}U_{,k} + U_{,jk}\dot{z}^k)r^j \\ &+ \delta_{ib}\delta_{jd}U_{,bd}\Omega_{ik}r^j r^k + O(\epsilon^5/R) \end{aligned} \tag{27}$$

As an immediate consequence of the two last equations, we deduce that

$$U_{,i}\dot{z}^i + U_{,4} \sim O(\epsilon^5/R)$$

and (28)

$$(U_{,j4} + \Omega_{jk}U_{,k} + U_{,jk}\dot{z}^k)r^j \sim O(\epsilon^5/R)$$

Since the last expression is satisfied for all r^j, we have

$$dU_{,i}/dt + \Omega_{ij}U_{,j} \sim O(\epsilon^5/R^2) \tag{29}$$

Expressions (28) and (29) agree with classical rigidity, to the effect that the Newtonian potential is constant along the line L and the Newtonian gravitational field is constant along the line L in the comoving system.

5. DISCUSSION

As pointed out in Section 3, L-rigidity is a specialization of pseudorigidity and thus is introduced choosing an arbitrary line L. For certain particular conditions, such as those in the relation between weak rigidity and L-rigidity, the center-of-mass line has been used in Minkowski space.

It is easy to prove that pseudorigidity, and thus L-rigidity, leads to classical rigidity if we identify the kinematical velocity ω^α with the dynamical velocity u^α (timelike proper vector of the energy-momentum tensor) in the post-Newtonian approximation:

$$v_a = \delta_{ai}\dot{z}^i - \delta_{ai}\Omega_{ij}r^j + O(\epsilon^3)$$

Moreover, the concept of L-rigidity includes a slight modification of the weak rigidity conditions. This modification does not imply that L-rigidity always coincides with weak rigidity, except for the very special class like the ones stated above.

We have just obtained that the baryon mass density is constant in first order, the stress tensor is constant in the comoving system, the Newtonian potential is constant along the line L, and the gravitational field is constant along the line L in the comoving system. These results would suggest that our concept of rigidity is more than just a pure specialization of pseudorigidity and a slight modification of weak rigidity.

On the other hand, let us indicate that the analysis of the L-rigidity conditions in the Newtonian approximation (Barreda and Olivert, 1993) would not be enough for our study, since, at this level of approximation, we get

$$\nabla \cdot \omega \sim O(\epsilon^3/R)$$

$$g(\mathfrak{L}_\omega N, N) \sim O(\epsilon^3/R)$$

and thus the first and the third conditions of L-rigidity are satisfied in this approximation order.

One may remark that in this paper we have not considered the relationship between this new concept of rigidity and that of quasirigidity. The authors are presently concluding this study and in subsequent work they will show that L-rigidity leads obviously to quasirigidity in very significant cases.

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